

# Brane-world evolution with brane-bulk energy exchange \*

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**ABSTRACT:** A rich variety of brane cosmologies is obtained once one allows for energy exchange between the brane and the bulk, depending on the precise form of energy transfer, on the equation of state of matter on the brane and on the spatial topology. This is demonstrated in the context of a non-factorizable background geometry with zero effective cosmological constant on the brane. An accelerating era is generically a feature of these solutions. In the case of low-density flat universe more dark matter than in the conventional FRW picture is predicted, while spatially compact solutions are found to delay their re-collapse. In addition to the above, which the interested reader will find in greater detail in [1], a first attempt towards a complete description of the full dynamics of both the bulk and the brane is reported.

## 1. Introduction

The idea that we might be living inside a defect, embedded in a higher dimensional space has already a long history. Concerning the nature of the defect, a solitonic codimension one or higher topological object was proposed [2] in the context of an ordinary higher dimensional gauge field theory, coupled [3] or not to gravity. It was soon realized, however, that, in contrast to scalar and spin-1/2 fields, it would be difficult to confine gauge fields on such an object. Various interesting ideas and scenaria were studied [4], which have not yet given a fully satisfactory picture. In connection with the topology and the size of the bulk space on the other hand, the popular choice was that the extra dimensions are compact, with size of order  $\mathcal{O}(M_{Pl}^{-1})$ . It was argued in [5], that in the context of the heterotic string with supersymmetry broken à la Scherk-Schwarz this should not be true anymore. The scale of supersymmetry breaking is tied to the size of internal dimensions, and a desirable

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\*Talk based essentially on [1]

supersymmetry breaking scale of a few TeV implies an extra dimension of about  $10^{-16}$  cm. Despite difficulties to build a realistic model based on these ideas, the scenario was taken seriously and analyzed further for its phenomenological consequences [6].

The situation is drastically different in the context of type-I string theory [7]. A few developments led to an exciting possibility and renewed interest in the whole idea. First, with the discovery of the D-branes as an essential part of the "spectrum" in type-I string theory, one could conjecture that we inhabit such a D-brane embedded in a ten-dimensional bulk. The usual solitonic defect of field theory was thus replaced by an appropriate collection of D-branes, which by construction confine the gauge fields [8], together with all the ingredients of the standard model. All known matter and forces lie on our brane world [9, 10], with the exception of gravity, which acts in the bulk space as well. It was, however, pointed out [11], that for Kaluza-Klein extra dimensions the gravitational force on the brane was consistent with all laboratory and astrophysical experimental data, as long as the extra dimensions were smaller than a characteristic scale. This led to the exciting possibility of two extra dimensions in the sub-millimeter range. Furthermore, it was demonstrated in the context of an appropriate effective five-dimensional theory of gravity, that once we take into account the back reaction of the brane energy-momentum onto the geometry of space-time, the graviton is effectively confined on the brane and Newton's law is reproduced to an excellent accuracy at large distances, even with a non-compact extra dimension [12].

At the same time, the analysis of the cosmological consequences of the above hypotheses attracted considerable interest. The first step was taken in [13], where the evolution of perfect fluid matter on the brane was studied, with no reference to the bulk dynamics and, therefore, no energy transfer between the brane and the bulk. Alternatively, a bulk-based point of view was adopted in [14], where the cosmology induced on a moving 3-brane in a static Schwarzschild-AdS<sub>5</sub> background was studied, and a general interpretation of cosmology on moving branes, together with the idea of "mirage" cosmology, were presented in [15]. The equivalence of the two approaches was demonstrated in [16]. It was also pointed out that branes provide natural mechanisms for a varying speed of light [17].

Energy-exchange between the brane and the bulk should in principle be included in any realistic cosmological scenario, and its effects have been studied in detail in the context of flat compact extra dimensions [18]. The role of energy-exchange on brane cosmology in the case of non-factorizable extra dimensions has not yet been investigated extensively [19, 20], even though the importance of energy outflow from the brane has already been demonstrated in the context of a Randall-Sundrum configuration with additional gravity induced on the brane [20].

Here an attempt towards a more complete analysis of the cosmic evolution of the brane in the presence of energy flow into or from the bulk is presented. The aim is to generalize the picture in bulk AdS (bulk gravity plus cosmological constant) by considering a general bulk theory (that includes gravity). There may be more bulk fields and more general bulk-brane couplings. As argued below, one may consider a regime where the bulk energy can be consistently neglected from the equations. Moreover, the energy exchange component of the energy-momentum tensor, will be assumed to be just a power of the matter density

of the brane. Although there are other valid parameterizations, the one chosen here is motivated by our previous work in [20], where we analyzed the out-flow of energy from the brane due to graviton radiation in the presence of induced gravity. Standard cross section calculations give an outflow that is a function of the temperature and other fundamental constants of the theory. If one re-expresses the temperature in terms of the running density using the cosmological equations we obtain a rate of flow that is a power of the density, with a dimensionful coefficient that depends on fundamental constants and initial energy densities. This is valid typically for a whole era, that is, piecewise in the cosmological evolution.

In a given theory, with a specific bulk content and brane-bulk couplings, the exponent of the density as well as the coefficient are calculable functions of the coupling constants of the Lagrangian as well as initial densities. It is also obvious that if the bulk theory is approximately conformal the most general form of rate of outflow will be polynomial in the density. In any case, the analysis carried out here is "phenomenological", and the parameters used in the energy exchange function have not been derived from fundamental dynamics.

The presentation is organized in five sections of which this introduction is the first. In section 2 the framework of this talk is described and the approximations on the brane-bulk exchange are presented. The effective set of equations for the analysis of the brane cosmology are derived. Section 3 contains several interesting characteristic solutions of the brane cosmology. In Section 4 a few brief comments will be offered concerning the exact treatment of the influx/outflow equations, together with a couple of representative cosmological histories, plotted to show the rich variety one may expect in the context of this scenario. Section 6 contains a first attempt to describe the full dynamics of both the bulk and the brane. With appropriate perfect fluids in the bulk and on the brane one may describe the late time cosmological evolution of the brane-world. Our results are summarized and the prospects for further research along these lines are discussed in the final section.

## 2. The model

The basic assumption is that we live on a 3-brane embedded in a 4+1-dimensional bulk with the transverse dimension assumed non-compact. In particular, we shall be interested in the generic model described by the action

$$S = \int d^5x \sqrt{-g} (M^3 R - \Lambda + \mathcal{L}_B^{\text{mat}}) + \int d^4x \sqrt{-\hat{g}} (-V + \mathcal{L}_b^{\text{mat}}), \quad (2.1)$$

where  $R$  is the curvature scalar of the five-dimensional metric  $g_{AB}$ ,  $A, B = 0, 1, 2, 3, 5$ ,  $\Lambda$  is the bulk cosmological constant, and  $\hat{g}_{\alpha\beta}$ , with  $\alpha, \beta = 0, 1, 2, 3$ , is the induced metric on the 3-brane. We identify  $(x, z)$  with  $(x, -z)$ , where  $z \equiv x_5$ . However, following the conventions of [12] we extend the bulk integration over the entire interval  $(-\infty, \infty)$ . The quantity  $V$  includes the brane tension as well as quantum contributions to the four-dimensional cosmological constant.  $\mathcal{L}_b^{\text{mat}}$  represents the matter on the brane. It includes massless

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excitations (branons) related to fluctuations of the brane, which is assumed frozen at  $z = 0$ . Finally,  $\mathcal{L}_B^{\text{mat}}$  stands for the bulk matter action.

As explained in the introduction, various instances of this model have been extensively discussed in the literature. Here we shall present some aspects of the influence of energy exchange between the brane and the bulk upon the cosmological evolution of our world on the brane.

We consider an ansatz for the metric of the form

$$ds^2 = -n^2(t, z)dt^2 + a^2(t, z)\gamma_{ij}dx^i dx^j + b^2(t, z)dz^2, \quad (2.2)$$

where  $\gamma_{ij}$  is a maximally symmetric 3-dimensional metric. We use  $k = -1, 0, 1$  to parameterize the spatial curvature.

The non-zero components of the five-dimensional Einstein tensor are

$$G_{00} = 3 \left\{ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} \right) - \frac{n^2}{b^2} \left( \frac{a''}{a} + \frac{a'}{a} \left( \frac{a'}{a} - \frac{b'}{b} \right) \right) + k \frac{n^2}{a^2} \right\}, \quad (2.3)$$

$$G_{ij} = \frac{a^2}{b^2} \gamma_{ij} \left\{ \frac{a'}{a} \left( \frac{a'}{a} + 2 \frac{n'}{n} \right) - \frac{b'}{b} \left( \frac{n'}{n} + 2 \frac{a'}{a} \right) + 2 \frac{a''}{a} + \frac{n''}{n} \right\} + \frac{a^2}{n^2} \gamma_{ij} \left\{ \frac{\dot{a}}{a} \left( -\frac{\dot{a}}{a} + 2 \frac{\dot{n}}{n} \right) - 2 \frac{\ddot{a}}{a} + \frac{\dot{b}}{b} \left( -2 \frac{\dot{a}}{a} + \frac{\dot{n}}{n} \right) - \frac{\ddot{b}}{b} \right\} - k \gamma_{ij}, \quad (2.4)$$

$$G_{05} = 3 \left( \frac{n'}{n} \frac{\dot{a}}{a} + \frac{a'}{a} \frac{\dot{b}}{b} - \frac{\dot{a}'}{a} \right), \quad (2.5)$$

$$G_{55} = 3 \left\{ \frac{a'}{a} \left( \frac{a'}{a} + \frac{n'}{n} \right) - \frac{b^2}{n^2} \left( \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{\ddot{a}}{a} \right) - k \frac{b^2}{a^2} \right\}. \quad (2.6)$$

Primes indicate derivatives with respect to  $z$ , while dots derivatives with respect to  $t$ .

The five-dimensional Einstein equations take the usual form

$$G_{AC} = \frac{1}{2M^3} T_{AC}, \quad (2.7)$$

where  $T_{AC}$  denotes the total energy-momentum tensor.

Assuming a perfect fluid on the brane and, possibly an additional energy-momentum  $T_C^A|_{m,B}$  in the bulk, we write

$$T_C^A = T_C^A|_{v,b} + T_C^A|_{m,b} + T_C^A|_{v,B} + T_C^A|_{m,B} \quad (2.8)$$

$$T_C^A|_{v,b} = \frac{\delta(z)}{b} \text{diag}(-V, -V, -V, -V, 0) \quad (2.9)$$

$$T_C^A|_{v,B} = \text{diag}(-\Lambda, -\Lambda, -\Lambda, -\Lambda, -\Lambda) \quad (2.10)$$

$$T_C^A|_{m,b} = \frac{\delta(z)}{b} \text{diag}(-\rho, p, p, p, 0), \quad (2.11)$$

where  $\rho$  and  $p$  are the energy density and pressure on the brane, respectively. The behavior of  $T_C^A|_{m,B}$  is in general complicated in the presence of flows, but we do not have to specify it further at this point.

We wish to solve the Einstein equations at the location of the brane. We indicate by the subscript o the value of various quantities on the brane. Integrating equations (2.3), (2.4) with respect to  $z$  around  $z = 0$  gives the known jump conditions

$$a'_{o+} = -a'_{o-} = -\frac{1}{12M^3}b_o a_o (V + \rho) \quad (2.12)$$

$$n'_{o+} = -n'_{o-} = \frac{1}{12M^3}b_o n_o (-V + 2\rho + 3p). \quad (2.13)$$

The other two Einstein equations (2.5) and (2.6) give

$$\frac{n'_o}{n_o} \frac{\dot{a}_o}{a_o} + \frac{a'_o}{a_o} \frac{\dot{b}_o}{b_o} - \frac{\dot{a}'_o}{a_o} = \frac{1}{6M^3}T_{05} \quad (2.14)$$

$$\frac{a'_o}{a_o} \left( \frac{a'_o}{a_o} + \frac{n'_o}{n_o} \right) - \frac{b_o^2}{n_o^2} \left( \frac{\dot{a}_o}{a_o} \left( \frac{\dot{a}_o}{a_o} - \frac{\dot{n}_o}{n_o} \right) + \frac{\ddot{a}_o}{a_o} \right) - k \frac{b_o^2}{a_o^2} = -\frac{1}{6M^3}\Lambda b_o^2 + \frac{1}{6M^3}T_{55}, \quad (2.15)$$

where  $T_{05}, T_{55}$  are the 05 and 55 components of  $T_{AC}|_{m,B}$  evaluated on the brane. Substituting (2.12), (2.13) in equations (2.14), (2.15) one obtains

$$\dot{\rho} + 3\frac{\dot{a}_o}{a_o}(\rho + p) = -\frac{2n_o^2}{b_o}T_5^0 \quad (2.16)$$

$$\begin{aligned} \frac{1}{n_o^2} \left( \frac{\ddot{a}_o}{a_o} + \left( \frac{\dot{a}_o}{a_o} \right)^2 - \frac{\dot{a}_o}{a_o} \frac{\dot{n}_o}{n_o} \right) + \frac{k}{a_o^2} &= \frac{1}{6M^3} \left( \Lambda + \frac{1}{12M^3}V^2 \right) - \\ &- \frac{1}{144M^6} (V(3p - \rho) + \rho(3p + \rho)) - \frac{1}{6M^3}T_5^5. \end{aligned} \quad (2.17)$$

We are interested in a model that reduces to the Randall-Sundrum vacuum [12] in the absence of matter. In this case, the first term on the right hand side of equation (2.17) vanishes. A new scale  $k_{RS}$  is defined through the relations  $V = -\Lambda/k_{RS} = 12M^3k_{RS}$ .

We shall further *assume* that the energy exchange between the bulk and the brane has negligible effect on the bulk. With this assumption, which is equivalent to assuming that the brane is embedded in a "heat reservoir", we shall be able to derive a solution of the brane cosmology that is independent of the bulk dynamics.

At this point we find it convenient to employ a coordinate frame in which  $b_o = n_o = 1$  in the above equations. This can be achieved by using Gauss normal coordinates with  $b(t, z) = 1$ , and by going to the temporal gauge on the brane with  $n_o = 1$ . The assumptions for the form of the energy-momentum tensor are then specific to this frame.<sup>1</sup>

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<sup>1</sup>If the vacuum energy dominates over the matter content of the bulk, we expect that the form of the metric will be close to the Randall-Sundrum solution with a static bulk. Thus, we expect (even though we cannot demonstrate explicitly without a full solution in the bulk) that in a generic frame, in which

$$\left| \frac{T_{m,B}^{\text{diag}}}{T_{v,B}^{\text{diag}}} \right| \ll 1 \quad (2.18)$$

we shall have  $\dot{b} \simeq 0$ . Then the transformation that sets  $b = 1$  is not expected to modify significantly the energy-momentum tensor.

Using  $\beta \equiv M^{-6}/144$  and  $\gamma \equiv VM^{-6}/144$ , and omitting the subscript o for convenience in the following, we rewrite equations (2.16) and (2.17) in the equivalent form

$$\dot{\rho} + 3(1+w) \frac{\dot{a}}{a} \rho = -T \quad (2.19)$$

$$\frac{\dot{a}^2}{a^2} = \beta \rho^2 + 2\gamma \rho - \frac{k}{a^2} + \chi \quad (2.20)$$

$$\dot{\chi} + 4 \frac{\dot{a}}{a} \chi = 2\beta \left( \rho + \frac{\gamma}{\beta} \right) T, \quad (2.21)$$

where  $p = w\rho$ ,  $T = 2T_5^0$  is the discontinuity of the zero-five component of the bulk energy-momentum tensor. As mentioned above, with an appropriate choice of parameters we have set to zero the effective cosmological constant  $\lambda = (\Lambda + V^2/12M^3)/12M^3$  on the brane, which should otherwise be added in the right-hand side of (2.20).

Equations (2.19), (2.20) and (2.21), in place of the usual FRW ones, supplemented by a set of initial conditions  $(\rho_i, a_i, \chi_i)$  or, equivalently,  $(\rho_i, a_i, \dot{a}_i)$ <sup>2</sup>, describe the dynamical evolution of matter on the brane, within the assumptions stated above. They have a straightforward interpretation. The energy density evolution equation (2.19) is modified by the presence of the energy exchange term in the right-hand side. Eq. (2.20) is the modification of Friedman equation. It has the extra term quadratic in the energy density, and may be thought of as the *definition* of the auxiliary density  $\chi$ . With this definition the other two equations are equivalent to the original system (2.16), (2.17). As we will see later on, in the special case of no-exchange ( $T = 0$ )  $\chi$  represents the mirage radiation reflecting the non-zero Weyl tensor of the bulk.

The second order equation (2.17) for the scale factor becomes

$$\frac{\ddot{a}}{a} = -(2+3w)\beta\rho^2 - (1+3w)\gamma\rho - \chi. \quad (2.22)$$

Notice that in the special case of  $w = 1/3$  one may define a new function  $\tilde{\chi} \equiv \chi + 2\gamma\rho$ . The functions  $\tilde{\chi}, \rho$  and  $a$  satisfy equations (2.19) to (2.22) with  $\tilde{\chi}$  in place of  $\chi$  and  $\gamma = 0$ . This should be expected, since for  $w = 1/3$  there is no  $\gamma$  left in equation (2.17).

It is not surprising that the introduction of the bulk and its interaction with the brane enriches the possible cosmologies on the latter. The interpretation of observations on the brane becomes unavoidably more complicated, as a consequence of the at best indirect knowledge of the evolution of the bulk and of its interaction with brane matter. This will become clear in the discussion of possible solutions to which we turn next.

### 3. Special solutions

Before we embark on the discussion of the general solutions of equations (2.19)–(2.21), it is instructive to start with a few special cases easy to treat analytically and whose physical

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<sup>2</sup>In the present case  $(\rho_i, a_i)$  of conventional FRW cosmology are not enough. The reason is that the generalized Friedmann equation (2.20) (or (3.2)) is not a first integral of the Einstein equations because of the possible energy exchange between the brane and the bulk.

content is more transparent. The analysis simplifies considerably in the low-density region, defined by  $\rho \ll \gamma/\beta = V$ , the case in which the matter energy density on the brane is much smaller than the brane cosmological term. In this case one may ignore the term  $\beta\rho^2$  in the above equations compared to  $\gamma\rho$ . As a result, equations (2.19)–(2.21) reduce to

$$\dot{\rho} + 3(1+w)H\rho = -T \quad (3.1)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = 2\gamma\rho + \chi - \frac{k}{a^2} \quad (3.2)$$

$$\dot{\chi} + 4H\chi = 2\gamma T. \quad (3.3)$$

### 3.1 Constant acceleration with $w \neq -1$

An interesting feature of this framework is the possible presence of accelerating cosmological solutions with constant acceleration, even for  $w \neq -1$ , that would be required in the context of the standard FRW cosmology. We can have exponential expansion with constant Hubble parameter  $H \equiv H_*$ , even if the brane content is not pure vacuum energy.

Indeed, let us choose for simplicity  $k = 0$  and search for a solution of equations (3.1)–(3.3) with constant  $H = H_*$ . It is straightforward to show that in this case the density  $\rho$  and the auxiliary field  $\chi$  are also constant<sup>3</sup>.

Denote with a \* subscript the constant values of these quantities. They satisfy

$$3H(1+w)\rho_* = -T(\rho_*) \quad (3.4)$$

$$H_*^2 = 2\gamma\rho_* + \chi_* \quad (3.5)$$

$$2H_*\chi_* = \gamma T(\rho_*). \quad (3.6)$$

It is clear from equation (3.4) that, for positive matter density on the brane ( $\rho > 0$ ), flow of energy into the brane ( $T(\rho) < 0$ ) is necessary. The accretion of energy from the bulk depends on the dynamical mechanism that localizes particles on the brane. It is not difficult to imagine scenaria that would lead to accretion. If the brane initially has very low energy density, energy can be transferred onto it by bulk particles such as gravitons. An equilibrium is expected to set in if the brane energy density reaches a limiting value. As a result, a physically motivated behavior for the function  $T(\rho)$  is to be negative for small  $\rho$  and cross zero towards positive values for larger densities. In the case of accretion it is also natural to expect that the energy transfer approaches a negative constant value for  $\rho \rightarrow 0$ .

The solution of equations (3.4)–(3.6) satisfies

$$T(\rho_*) = -\frac{3\sqrt{\gamma}}{\sqrt{2}}(1+w)(1-3w)^{1/2}\rho_*^{3/2} \quad (3.7)$$

$$H_*^2 = \frac{1-3w}{2}\gamma\rho_* \quad (3.8)$$

$$\chi_* = -\frac{3(1+w)}{2}\gamma\rho_*. \quad (3.9)$$

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<sup>3</sup>Combining (3.1) and (3.3) one obtains  $\dot{\chi} + 2\gamma\dot{\rho} + 6\gamma(1+w)H_*\rho + 4H_*\chi = 0$ , which, combined with (3.2) gives  $\chi = -2\rho/(3\gamma(1+w))$ . This, upon substitution into (3.2) leads to  $\rho = \text{constant}$ .

What is the interpretation of this result? For a general form of  $T(\rho)$ , determined by the physics of fundamental interactions, equation (3.7) is an algebraic equation with a discrete number of roots. For any value of  $w$  in the region  $-1 < w < 1/3$  a solution is possible. The resulting cosmological model is characterized by a scale factor that grows exponentially with time. The energy density on the brane remains constant due to the energy influx from the bulk. The model is very similar to the steady state model of cosmology [?]. The main differences are that the energy density is not spontaneously generated, and the Hubble parameter receives an additional contribution from the "mirage" field  $\chi$  (see equation (3.5)).

By linearizing equations (3.1)–(3.3) around the above solution, one may study its stability. The conclusion is that for  $-1 < w < 1/3$  and  $0 < \tilde{\nu} < 3/2$ , where  $\tilde{\nu} \equiv d \ln |T|(\rho_*)/d \ln \rho$ , the fixed point solution described here is stable [1].

For  $w = -1$  we get the standard inflation only for a value  $\rho_*$  that is a zero of  $T(\rho)$ . In this case there is no flow along the fifth dimension and also  $\chi_* = 0$ .

### 3.2 "Mirage" radiation for energy outflow

Another interesting situation arises in the case of outflow  $T > 0$ , of energy from the brane into the bulk and with radiation domination on the brane, i.e. with  $p = \rho/3$ <sup>4</sup>. Equations (3.1) and (3.3) have an exact solution independent of the explicit form of  $T$ :

$$\rho + \frac{\chi}{2\gamma} = \left( \rho_i + \frac{\chi_i}{2\gamma} \right) \frac{a_i^4}{a^4} \quad (3.10)$$

and

$$H^2 = (2\gamma\rho_i + \chi_i) \frac{a_i^4}{a^4} - \frac{k}{a^2}. \quad (3.11)$$

Assume that initially  $\chi_i = 0$ . It is clear that the effect of the radiation on the expansion does not disappear even if it decays during the cosmological evolution: the Hubble parameter of equation (3.11) is determined by the initial value of the energy density, diluted by the expansion in a radiation dominated universe. The real radiation energy density  $\rho$ , however, falls with time faster than  $a^{-4}$ .

To appreciate the unusual features of the resulting cosmology, consider the simple case of what we shall call "radioactive" matter on the brane,  $T = A\rho$  with  $A > 0$ . Then equation (3.1) can be integrated for arbitrary  $w$  to give

$$\rho = \rho_i \left( \frac{a_i}{a} \right)^{3(1+w)} e^{-At}, \quad (3.12)$$

where we have considered the general case  $p = w\rho$ . This equation has an obvious interpretation. The energy density on the brane dilutes both as a result of the expansion of the universe (the  $a^{-3(1+w)}$  factor) as well as a consequence of its "radioactive decay" (the  $e^{-At}$

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<sup>4</sup>This is the simplest case to treat analytically. The conclusions of this subsection generalize [1] to other interesting cases, such as dust on the brane.

factor). In the special case  $w = 1/3$  we are considering here, one obtains <sup>5</sup>

$$\rho = \rho_i \frac{a_i^4}{a^4} e^{-At}, \quad (3.13)$$

$$\chi = 2\gamma\rho_i \frac{a_i^4}{a^4} (1 - e^{-At}). \quad (3.14)$$

Imagine an initial energy density  $\rho_i \neq 0$ , while  $\chi_i = 0$ . The Hubble parameter, given by equation (3.11) with  $\chi_i = 0$ , corresponds to an initial radiation density  $\rho_i$ , further diluted only by the expansion. At late times the energy density disappears exponentially fast and the expansion is the consequence of a "mirage" effect, sustained, through the auxiliary  $\chi$ , by the expansion itself. Of course, the presence of a "mirage" term is possible even without energy flow, since equation (3.3) has a solution  $\chi = C/a^4$  even for  $T = 0$ , and  $\chi$  can act as "mirage" radiation. The novel feature for  $T \neq 0$  is that this "mirage" effect appears through the decay of real brane matter, even if it was absent in the beginning ( $\chi_i = 0$ ).

### 3.3 The case of radiation for energy influx

In the case of radiation the general solution of equations (3.1)–(3.3) was derived in the previous subsection and is given by equations (3.10), (3.11). The expansion is that of a radiation-dominated universe with constant energy  $(\chi_i/(2\gamma) + \rho_i)a_i^4$  per co-moving volume. The "mirage" energy density is diluted  $\sim a^{-4}$ .

The explicit dependence on time will be discussed next in the case of flat space ( $k = 0$ ), in which the energy density satisfies

$$\frac{d\rho}{dt} + \frac{2}{t}\rho = -T(\rho). \quad (3.15)$$

If  $T(\rho) < 0$  for all  $\rho$ , and the "friction" term in the left hand side becomes suppressed for  $t \rightarrow \infty$ , we expect an unbounded increase of  $\rho$  in this limit. For  $\rho \gtrsim \gamma/\beta$  the low energy approximation employed in this section breaks down. The full treatment necessary in this case will be given in the next section.

The actual situation is rather complicated and the details depend crucially on the form of  $T(\rho)$ . Assuming that  $T(\rho) = A\rho^\nu$  with  $A < 0$ , the exact solution of equation (3.15) for  $\nu \neq 1, 3/2$  is

$$(\rho\tilde{t}^2)^{1-\nu} = (\rho_i\tilde{t}_i^2)^{1-\nu} + \frac{1-\nu}{3-2\nu} (\tilde{t}^{3-2\nu} - \tilde{t}_i^{3-2\nu}), \quad (3.16)$$

where  $\tilde{t} = |A|t$ . For  $\nu = 1$  the solution is

$$\rho = \rho_i \frac{\tilde{t}_i^2}{\tilde{t}^2} e^{\tilde{t} - \tilde{t}_i}, \quad (3.17)$$

and for  $\nu = 3/2$

$$(\rho\tilde{t}^2)^{-1/2} = (\rho_i\tilde{t}_i^2)^{-1/2} - \frac{1}{2} \ln \left( \frac{\tilde{t}}{\tilde{t}_i} \right). \quad (3.18)$$

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<sup>5</sup>Having neglected the term  $\beta\rho^2$  from equations (3.2) and (3.3), these solutions are valid only if  $\beta\rho_i^2 \ll H_i^2$ . As we shall point out below this condition is eventually satisfied for all solutions in the case of outflow of the form discussed here and  $k = 0$ .

For  $0 \leq \nu < 1$  we have  $\rho \sim \tilde{t}^{1/(1-\nu)}$  for  $\tilde{t} \rightarrow \infty$ . For  $\nu = 1$  the increase of the energy density at large  $\tilde{t}$  is exponential moderated by a power. For  $1 < \nu < 3/2$  the energy density diverges at a finite time

$$\tilde{t}_d^{3-2\nu} = \tilde{t}_i^{3-2\nu} + \frac{3-2\nu}{1-\nu} (\rho_i \tilde{t}_i^2)^{1-\nu}. \quad (3.19)$$

A similar divergence appears for  $\nu = 3/2$ . For  $\nu > 3/2$  a divergence occurs if the quantity

$$D = (\rho_i \tilde{t}_i^2)^{1-\nu} - \frac{\nu-1}{2\nu-3} \tilde{t}_i^{2\nu-3} \quad (3.20)$$

is negative. In the opposite case  $\rho \tilde{t}^2 \rightarrow 1/D$  for  $t \rightarrow \infty$ , and the energy density diminishes:  $\rho \sim t^{-2}$ .

As we discussed earlier, it is physically reasonable that the energy influx should stop at a certain value  $\rho_{cr}$ , and be reversed for larger energy densities. The dynamical mechanism that localizes particles on the brane cannot operate for arbitrarily large energy densities. This modifies the solutions above that predict an unbounded increase of the energy density.

A final observation that will be encountered again in the next section is that, despite of the fact that the energy density in most cases increases for large times, it can decrease at the initial stages. This is obvious from eq. (3.15). If at the time  $t_r$  that the brane enters a radiation dominated era  $|T(\rho)|/\rho < 2/t$ , the energy density decreases for a certain time.

### 3.4 Non-flat solutions

In addition to the analytical special solutions discussed above, we would like to present a few suggestive numerical results concerning the  $k = \pm 1$  cases. For  $\nu = 1$ , we substitute (3.12), true for any  $A$ , into the second order equation (2.17), to obtain

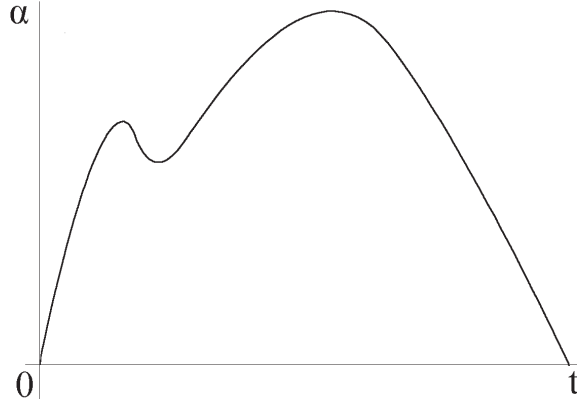
$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} + (1+3w)\beta\rho_1^2 \frac{e^{-2At}}{a^{6(1+w)}} - (1-3w)\gamma\rho_1 \frac{e^{-At}}{a^{3(1+w)}} = 0. \quad (3.21)$$

It is obvious from (3.21) that for outflow with  $k = 1$  and  $w \geq -1/3$  the Universe will exhibit eternal deceleration. In particular, in the case of dust, figure 1 depicts the solution for  $a(t)$  of (3.21) for some initial conditions for  $a, \rho$ . Notice that  $a(t)$  after a period of decrease, starts increasing again, thus delaying its re-collapse. Of course, with appropriate initial conditions one also obtains solutions with the standard FRW behavior.

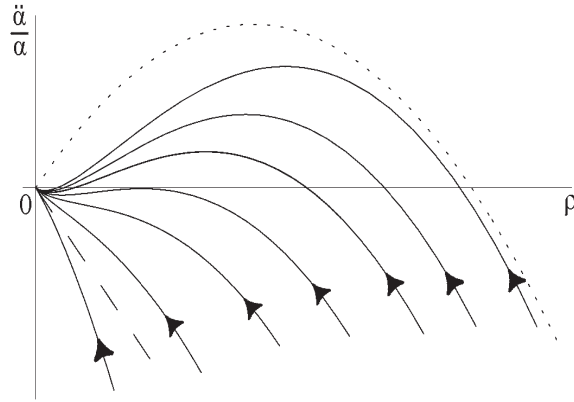
For an open ( $k = -1$ ) Universe with  $A > 0$ ,  $w = 0$ ,  $\nu = 1$  we have found numerically a solution where  $\rho(t)$  monotonically decreases to zero, while  $a(t)$  starts with deceleration, but later on accelerates eternally; more specifically,  $q(t) \rightarrow 0^+$  for  $t \rightarrow 0$ . Another possibility for  $k = -1$  and  $A < 0$ ,  $w = 0$ ,  $\nu = 1$  allows for a Universe starting with acceleration at infinite densities, which later on, turns to deceleration with  $\rho$  approaching a constant value.

## 4. General remarks on the outflow/influx equations

In [1] the interested reader may find a rather detailed study of the fundamental equations of cosmology (2.19)–(2.21) in the particular context presented here.



**Figure 1:** Outflow,  $k = +1$ ,  $w = 0$ ,  $\nu = 1$ .



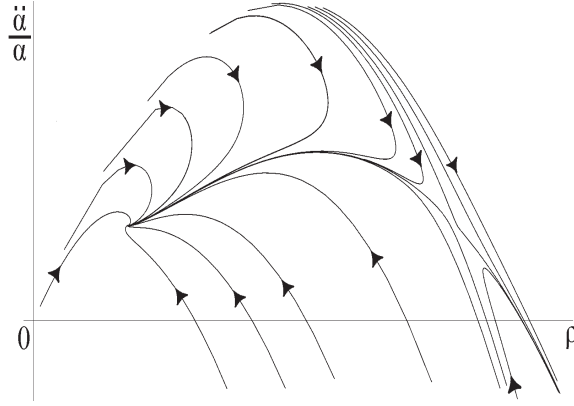
**Figure 2:** Outflow,  $k = 0$ ,  $w = 0$ ,  $\nu = 1$ . The arrows show the direction of increasing scale factor.

The general analysis, however, was done numerically for various cases of the energy exchange functional dependence on the density  $\rho$ , various values of  $k$  and various types of matter on the brane. Two examples with of the resulting cosmologies are the ones shown on Figures 2 and 3.

Figure 2 shows the global phase portrait of  $q \equiv \ddot{a}/a$  versus  $\rho$  for the case of dust on a flat ( $k = 0$ ) brane, with energy *outflow* and  $T = A\rho$ . The arrows point in the direction of increasing time, or equivalently of increasing scale factor for the expanding solutions we are interested in. In all cases the density decreases with time and the universe evolves towards the fixed point with vanishing energy density and zero  $q$ . One may, however, recognize two classes of evolutions. The one in which the universe decelerates at all times, and the one in which the universe first decelerates, then it lives a period of acceleration before it decelerates again to approach the fixed point from below the  $\rho$  axis. The behavior near the fixed point may be derived analytically. Also analytically one may derive the presence of the limiting parabola  $q = \gamma\rho - \beta\rho^2$ , shown with a dotted line. For comparison, we also show the straight dashed line  $q = -(1 + 3w)\gamma\rho$ , which represents the standard FRW behavior.

A lot richer and diverse is the case of energy *influx*. The cosmological evolution of dust on a flat brane with linear exchange function  $T(\rho)$  is presented in Figure 3. Again,

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**Figure 3:** Influx,  $k = 0$ ,  $w = 0$ ,  $\nu = 1$ .

as in the previous case, one may recognize the limiting parabola, restricting all possible histories of the brane. Its slope at the origin is also determined analytically and verified numerically.

Not shown is the line  $q = -(1 + 3w)\gamma\rho$  of FRW cosmology. Due to the energy influx, the origin is not anymore a future fixed point. Instead, there are two fixed points, both with constant acceleration and constant energy density. The one on the left is an attractive fixed point for almost all initial conditions. The other is a saddle point corresponding to unstable trajectories. Clearly, there is now a variety of possible histories, but it is premature to concentrate on any one of them in an effort to understand our Universe. One has to include the dynamics of the bulk and solve the complete system, before such an attempt can be anything but academic.

## 5. A first step towards a complete description of the brane-world cosmology.

The analysis so far was based on the assumption that the bulk is some kind of reservoir and does not change appreciably during the cosmological evolution of the brane. So, the bulk was neglected and the autonomous system of equations for the brane and the energy exchange with the bulk was discussed. However, realistic applications of this scenario call for a more general treatment, including the dynamics of the bulk.

A first step towards a complete analysis of the brane-bulk coupled system, is to try at least to describe the stable fixed point of the influx case, the far future behavior of the brane-world.

It is straightforward to convince oneself that the metric

$$ds^2 = -dt^2 + e^{bt+c|z|} dx^i dx^i + dz^2 \quad (5.1)$$

satisfies the five-dimensional Einstein equations with energy-momentum tensor, whose non-vanishing components are ( $8\pi G = 1$ )

$$T_0^0 = \frac{3}{4}(2c^2 - b^2) + 3c\delta(z) \quad (5.2)$$

$$T_1^1 = T_2^2 = T_3^3 = \frac{3}{4}(c^2 - b^2) + 2c\delta(z) \quad (5.3)$$

$$T_5^5 = \frac{3}{4}(c^2 - 2b^2) \quad (5.4)$$

$$T_5^0 = \frac{3}{2}bc\epsilon(z) \quad (5.5)$$

To decide about the nature of matter in the bulk, one has to transform to a coordinate frame with vanishing  $T_5^0$ . This is achieved by a Lorentz transformation in the  $t-z$  subspace. Such a transformation will leave the  $-dt^2 + dz^2$  part of the metric invariant, while the exponent in the metric will depend linearly on only one of the new coordinates. Specifically, one distinguishes two cases.

Case A.  $|c| < |b|$ . The transformed metric takes the form

$$ds^2 = -dt'^2 + e^{b\sqrt{1-c^2/b^2}t'} dx^i dx^i + dz'^2 \quad (5.6)$$

with the corresponding energy-momentum tensor

$$T'^\mu_\nu = -\frac{3}{4}(b^2 - c^2) \text{diag}(1, 1, 1, 1, 2). \quad (5.7)$$

One interpretation is that in the bulk there exists a perfect fluid with constant energy density and pressure, given by

$$\rho_B = \frac{3}{4}(b^2 - c^2) > 0, P_B = -\frac{3}{4}(b^2 - c^2) = -\rho_B, P_T = \frac{3}{2}(c^2 - b^2) = -2\rho_B \quad (5.8)$$

flowing towards the brane in the brane rest frame.

Case B.  $|c| > |b|$ . The transformed metric takes the form

$$ds^2 = -dt''^2 + e^{c\sqrt{1-b^2/c^2}z''} dx^i dx^i + dz''^2 \quad (5.9)$$

with the corresponding energy-momentum tensor

$$T''^\mu_\nu = -\frac{3}{4}(b^2 - c^2) \text{diag}(2, 1, 1, 1, 1). \quad (5.10)$$

interpreted as a perfect fluid with constant energy density and pressure, given by

$$\rho_B = \frac{3}{2}(b^2 - c^2) < 0, P_B = -\frac{3}{4}(b^2 - c^2) = -\frac{1}{2}\rho_B, P_T = -\frac{3}{4}(b^2 - c^2) = -\frac{1}{2}\rho_B \quad (5.11)$$

again flowing towards the brane in the brane rest frame.

The induced metric on the brane is just

$$d\tilde{s}^2 = -dt^2 + e^{bt} dx^i dx^i, \quad (5.12)$$

which for  $b > 0$  is an exponentially expanding deSitter space-time with constant  $q$ .

In addition, on the brane there is a perfect fluid again with constant energy density and pressure

$$\rho_b = -3c, p_b = 2c = -2\rho_b/3. \quad (5.13)$$

For  $c < 0$ , to avoid exponential blow-up of the metric at transverse-space infinity, the brane density is positive. Assuming linear state equation for the matter on the brane, the situation corresponds to  $w_b = -2/3$ .

Finally, there is constant energy influx from the bulk. Indeed,  $T(z > 0) \equiv T_5^0(z > 0)/2 = 3bc/4 < 0$ , for the chosen signs of  $b$  and  $c$ .

It seems that the above metric provides a complete description of the brane-world cosmology at late times. It realises the main characteristics of the stable fixed point of Figure 3. It describes a brane carrying constant positive density, expanding with constant acceleration in the presence of constant influx from the bulk.

There are however a few unsatisfactory features in the above solution. First, instead of dust on the brane we had to introduce a fluid with  $w = -2/3$ . Second, assuming a linear equation of state for the matter in the bulk, the above solution of 5-dimensional Einstein gravity with a brane does not satisfy in the bulk any of the known energy conditions. Indeed, the perfect fluid in the bulk does not satisfy the null energy condition (NEC) ( $\rho_B + P_B \geq 0$  and  $\rho_B + P_T \geq 0$ ) and consequently, it does not satisfy any of the other known energy conditions (weak, strong, dominant or null dominant). Nevertheless, it should be pointed out in connection with these comments, that on the one hand violation of the energy conditions does not necessarily mean that the theory is fundamentally sick and unacceptable, by being acausal or unstable (Remember, for instance, the recent discussion in the literature of the so called "phantom" matter). On the other hand one does not have to interpret the above solution assuming a linear relation between pressure and density (see for instance [21]) and this might allow for an interpretation with more "natural" matter, both on the brane and in the bulk.

## 6. Conclusions

A rather detailed mathematical analysis of the role of the brane-bulk energy exchange on the evolution of a Brane Universe, adequate for a wide range of potentially realistic implementations, was presented. The effective brane cosmological equations were derived with perfect fluid matter on the brane, constant energy-momentum tensor in the bulk and non-vanishing exchange between them. A detailed study of the solutions of these unconventional equations was performed in the case of zero effective cosmological constant on the brane, in order to reduce to the Randall-Sundrum vacuum in the absence of matter. The analysis revealed a rather rich variety of possible cosmologies, depending upon the precise form of the exchange term, the topology of 3-space, and the nature of matter on the brane.

General characteristic properties of the solutions were presented, together with a few special but particularly interesting solutions, obtained in the limit of low energy density on the brane. One of these is the exactly solvable case of radiation, where a mirage radiation effect appears through the decay of real brane matter. Another, is a De Sitter fixed point solution, obtained in the case of energy influx, even without pure vacuum energy, which in addition is stable for a wide range of reasonable forms of energy exchange.

Clearly, the formulation of a detailed brane-world cosmological scenario requires a complete analysis of the full brane-bulk system. A first step towards understanding the full dynamics was taken above and a description of the asymptotic times cosmology was given. It has a positive brane density and a positive  $q$ , and represents nicely the attractive fixed point plotted in Figure 3, for the case of energy influx. Many open questions, such as an all-time description of the brane-world evolution, the duration of accelerating periods, the creation of primordial fluctuations, and the compatibility with conventional cosmology at low energy densities, should be addressed. However, we believe that the cosmological evolution in the context presented here has many novel features, that may provide answers to outstanding questions of modern cosmology.

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